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TOWARD THE PROBLEM OF THE RESONANCES OF A BUBBLE RADIALLY PULSATING IN A LIQUID

N. S. Khabeev

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Gas bubbles performing radial pulsations under the action of an acoustic field are analyzed. It is shown in detail that gas bubbles of constant mass may have only one resonance frequency.

Keywords: *resonance, bubble, pulsations, acoustic field.*

Introduction. This problem was considered in a number of works. It has been found that in the case of a vapor bubble in a boiling liquid, as well as of a soluble bubble, there exists a new resonance frequency that differs from Minnaert's. The indicated effect is undeniably connected with the presence of mass transfer. However, investigations are available (see, e.g., [1]) in which it is mistakenly stated that the presence of two resonance frequencies is typical also of a gas bubble with a constant mass in interphase heat exchange. No detailed analysis of [1] has been made by anybody, and the erroneous conclusion drawn there has not been convincingly refuted. To do away with the lack of understanding in this important problem, the work of [1] is analyzed in detail below, and the incorrectness of its principal conclusion is shown.

A simple formula for the frequency of free radial oscillations of an adiabatic gas bubble was obtained by Minnaert [2] for the first time:

$$\omega = \frac{1}{R_0}\sqrt{\frac{3\gamma p_0}{\rho_{\text{liq}}}} \,. \tag{1}$$

Later, in the course of the analysis of oscillating vapor bubbles in cryogenic fluids, it was found in [3] by means of numerical calculations that there was a second resonance frequency. The latter work contains some inaccuracies that, however, do not influence its correct inference. Thus, in particular, the resonance frequency was determined from the condition $\text{Re}\{S\} = 0$, where S is the denominator of the expression for the amplitude of the bubble radius fluctuations. Actually, the imaginary part of the resonance function S also depends on frequency. Therefore, the resonance frequency must be determined from the equation $\partial|S|\partial\omega = 0$. This is expounded in more detail in [4]. The inference on the existence of the second resonance frequency of a vapor bubble was confirmed by numerical calculations also in [5].

Attempts at obtaining a simple analytical formula that would relate the second resonance frequency of a bubble to its radius have been made. Such an attempt implemented, e.g., in [6], turned out to be erroneous, as noted in later works (see, e.g., [4, 7]). For the first time, a correct formula for the second resonance frequency of a vapor bubble was obtained in [8]. Later a relation of the same type was suggested in [9]. In [10], it was shown that a similar effect of the existence of two resonances is also typical of soluble gas bubbles in a liquid. In the latter work a simple formula that relates the second resonance frequency to the bubble radius was obtained.

Work [1] stands somewhat by itself; it is mistakenly stated there that the presence of two resonance frequencies is also characteristic for gas bubbles of constant mass in the presence of heat transfer. This leads to some misunderstanding about this problem. To remove this confusion, the work of [1] is analyzed in detail below.

Formulation of the Problem and Basic Equations. We consider a spherical gas bubble performing forced oscillations under the action of an acoustic field in a viscous liquid. We assume that the pressure in the bubble is homogeneous and that the process has a spherical symmetry. The pressure is homogeneous when the sonic wavelength in the gas is much larger than the bubble size.

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Within the framework of the assumptions made, the equations of heat influx and continuity for the gas phase in spherical Eulerian coordinates (r, t) have the form

$$\rho_g c_p \left(\frac{\partial T_g}{\partial t} + V_g \frac{\partial T_g}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\lambda_g r^2 \frac{\partial T_g}{\partial r} \right) + \frac{dp_g}{dt} , \quad \frac{\partial \rho_g}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_g V_g) = 0 . \tag{2}$$

As is shown in [11], the boundary condition for the temperature on the bubble surface can be adopted in the form

$$r = 0; \ \frac{\partial T}{\partial r} = 0 . \tag{3}$$

where T_0 is the liquid temperature far from the bubble. The liquid behaves like a thermostat. The boundary condition at the center of the bubble is

$$r = 0 : \ \frac{\partial T}{\partial r} = 0 . \tag{4}$$

The dynamics of the radial oscillations is described by the Rayleigh equation [12, 13]:

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 = \frac{p_g - p_\infty - \frac{2\sigma}{R}}{\rho_{liq}} - \frac{4\mu_{liq}}{\rho_{liq} R} \frac{dR}{dt} . \tag{5}$$

Equation (5) may include additional terms due to account for the liquid compressibility in a quasi-acoustic approximation. However, in the present problem these additional terms are insignificant.

The gas state equation is given as

$$p = \rho F T . \tag{6}$$

The condition of pressure homogeneity in the bubble allows one to obtain the following relation from Eqs. (2) and (6):

$$\frac{dp}{dt} = \frac{3(\gamma - 1)}{R} \lambda_g \left. \frac{\partial T_g}{\partial r} \right|_R - \frac{3\gamma p}{R} \frac{dR}{dt} . \tag{7}$$

In the case of small radial oscillations in an acoustic field the pressure far from the bubble can be described by the equation

$$p_{(\infty)} = p_\infty + p_a \exp(i\omega t) , \quad p_a << p_\infty , \tag{8}$$

where p_∞ is the static pressure in the liquid. Then the bubble radius will be represented by the real part of the expression

$$R = R_0 (1 + \alpha \exp(i\omega t)) , \tag{9}$$

where α is the dimensionless amplitude of the bubble oscillations: $|\alpha| << 1$ and ω is the frequency of oscillations. It is also assumed that the small deviations of the pressure and temperature in the bubble from the equilibrium state have the form [5]

$$p = p_0 [1 + P \exp(i\omega t)] , \quad T = T_0 [1 + \theta(r) \exp(i\omega t)] . \tag{10}$$

The system of basic equations can be linearized.

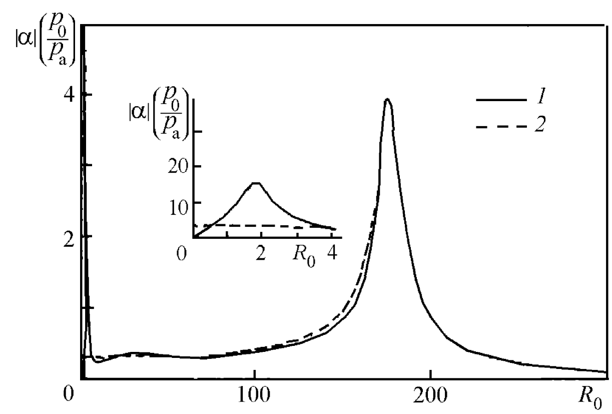

Fig. 1. The amplitude of a gas bubble oscillating in water at a frequency of 18 kHz vs. its radius [1]: 1, present theory; 2, [15].

For gas bubbles in a viscous liquid the solution obtained in [4, 14] for the amplitude of bubble radius fluctuations has the form $\alpha = p_a/S$. We will show the inconsistency of the curves in Fig. 1 that corresponds to Fig. 2 in [1] and of the conclusion drawn on their basis on the existence of the second resonance size of a gas bubble. In [1], the results of calculations are compared with the results of [15]. The inset in Fig. 1 contains the results of [15] that are unjustifiably criticized in [1]. We will consider the quantity $\beta_1 = p_a/(p_\infty \alpha)$ as a function of R_0. According to [1], in the notation adopted

$$\beta_1 = \frac{\rho_{\rm liq}\omega^2 R_0^2}{p_\infty\left(1 + \frac{i\omega R_0}{a_{\rm liq}}\right)} - \gamma \frac{1 + \frac{2\sigma}{R_0 p_\infty}}{\frac{1}{3} + \frac{\gamma - 1}{ix}\left[(ix)^{1/2}\coth(ix)^{1/2} - 1\right]} + \frac{2\sigma}{R_0 p_\infty} - \frac{4i\mu_{\rm liq}\omega}{p_\infty}, \quad x = \frac{\gamma\omega R_0^2}{D_{\rm g}}. \tag{11}$$

We will introduce a new variable:

$$y = \left(\frac{2\gamma\omega}{D_{\rm g}}\right)^{1/2} R_0 \tag{12}$$

and the notation

$$G = \left(\frac{D_{\rm g}}{2\gamma\omega}\right)^{1/2}, \quad A = \frac{\rho_{\rm liq}\omega D_{\rm g}}{2\gamma p_\infty}, \quad B = \frac{\omega G}{a_{\rm liq}}, \quad c = \frac{2\sigma}{p_\infty G}, \quad D = \frac{4\mu_{\rm liq}\omega}{p_\infty}. \tag{13}$$

We will estimate the order of the quantities in application to the variant depicted in Fig. 1 (the liquid is water at 20$^{\rm o}$C, the gas is air, $f = \omega/2\pi = 18$ kHz):

$$G \approx 9.3\ \mu{\rm m}, \quad A \approx 0.011, \quad B \approx 0.75\cdot 10^{-3}, \quad c \approx 0.16, \quad D \approx 0.0045. \tag{14}$$

In the new notation Eq. (11) can be rewritten as follows:

$$\beta_1 = \frac{Ay^2}{1 + iBy} - \gamma \frac{1 + \frac{c}{y}}{\frac{1}{3} + 2\frac{\gamma - 1}{iy^2}\left[\left(\frac{iy^2}{2}\right)^{1/2}\coth\left(\frac{iy^2}{2}\right)^{1/2} - 1\right]} + \frac{c}{y} - iD. \tag{15}$$

We avail ourselves of the equality

$$b^{1/2}\coth b^{1/2} = \frac{x}{\sqrt{2}}\,\frac{\sinh\sqrt{2x}+\sin\sqrt{2x}}{\cosh\sqrt{2x}-\cos\sqrt{2x}} + \frac{ix}{\sqrt{2}}\,\frac{\sinh\sqrt{2x}-\sin\sqrt{2x}}{\cosh\sqrt{2x}-\cos\sqrt{2x}}, \quad b = ix . \tag{16}$$

Then

$$\frac{2}{iy^2}\left[\left(\frac{iy^2}{2}\right)^{1/2}\coth\left(\frac{iy^2}{2}\right)^{1/2} - 1\right] = \frac{2}{iy^2}\left(-1 + \frac{y}{2}\,\frac{\sinh y+\sin y}{\cosh y-\cos y} + \frac{iy}{2}\,\frac{\sinh y-\sin y}{\cosh y-\cos y}\right)$$

$$= \frac{\sinh y-\sin y}{y\,(\cosh y-\cos y)} + i\left[\frac{2}{y^2} - \frac{\sinh y+\sin y}{y\,(\cosh y-\cos y)}\right]. \tag{17}$$

We will denote

$$X = \frac{3\,(\gamma-1)}{\gamma}\left[\frac{1}{3} - \frac{\sinh y-\sin y}{y\,(\cosh y-\cos y)}\right], \tag{18}$$

$$Z = \frac{3\,(\gamma-1)}{\gamma}\left[\frac{1}{3} - \frac{\sinh y+\sin y}{y\,(\cosh y-\cos y)} - \frac{2}{y^2}\right]. \tag{19}$$

Equation (15) can be rewritten in the form

$$\beta_1 = \frac{Ay^2}{1+iBy} - \frac{3\left(1+\frac{c}{y}\right)}{1-X-iZ} + \frac{c}{y} - iD . \tag{20}$$

We will single out the real and imaginary parts of β_1:

$$\beta_2 = -\frac{1}{i}\,\mathrm{Im}\,\beta_1 = D + \frac{ABy^3}{1+B^2y^2} + \frac{3\left(1+\frac{c}{y}\right)Z}{(1-X)^2+Z^2}, \tag{21}$$

$$\beta_3 = \mathrm{Re}\,\beta_1 = \frac{Ay^2}{1+B^2y^2} - \frac{3\left(1+\frac{c}{y}\right)(1-X)}{(1-X)^2+Z^2} + \frac{c}{y}. \tag{22}$$

Let us estimate the expression $\frac{1-X}{(1-X)^2+Z^2}$. From Eq. (18) we have

$$\frac{\gamma}{\gamma-1}X = 1 - \frac{3\,(\sinh y-\sin y)}{y\,(\cosh y-\cos y)} = 1 - \frac{3\left(\frac{y^3}{3!}+\frac{y^7}{7!}+\ldots\right)}{y\left(\frac{y^2}{2!}+\frac{y^6}{6!}+\ldots\right)} = \frac{y^4\left(\frac{2}{6!}-\frac{6}{7!}\right)+y^8\left(\frac{2}{10!}-\frac{6}{11!}\right)+\ldots}{1+2\,\frac{y^4}{6!}+2\,\frac{y^8}{10!}+\ldots}$$

$$= 2\left[y^4\,\frac{4}{7!} + \frac{y^8\left(\frac{11-3}{11!}-\frac{4}{7!}\,\frac{2}{6!}\right)+y^{12}\left(\frac{15-3}{15!}-\frac{4}{7!}\,\frac{2}{10!}\right)+\ldots}{1+\frac{2y^4}{6!}+\frac{2y^2}{10!}}\right] = \frac{8}{7!}\,y^4 - \varepsilon_1 , \tag{23}$$

where $\varepsilon_1 > 0$, since all the terms in the latter expression for the numerator of the fraction in (23) are negative. Next, we obtain

$$\frac{\gamma}{3(\gamma-1)} Z = \frac{y + \frac{y^5}{5!} \ldots}{y\left(\frac{y^2}{2!} + \frac{y^6}{6!} + \ldots\right)} - \frac{2}{y^2} = 2\,\frac{\left(\frac{1}{5!} - \frac{2}{6!}\right) y^2 + \left(\frac{1}{9!} - \frac{2}{10!}\right) y^6 + \ldots}{1 + \frac{2y^4}{6!} + \ldots} = \frac{8}{6!} y^2 - \varepsilon_2 , \tag{24}$$

similar to $\varepsilon_2 > 0$.

Since at $x > a$

$$\frac{d}{dx}\left(\frac{x}{x^2 + a^2}\right) = \frac{a^2 - x^2}{(x^2 + a^2)^2} < 0$$

and $X > 0$ and $Z > 0$ at any positive values of y, in the region of $X + Z < 1$ the following inequality is valid:

$$1 - Z^2 < \frac{1}{1 + Z^2} < \frac{1 - X}{(1 - X)^2 + Z^2} < \frac{1 - \frac{8}{7!}\frac{\gamma-1}{\gamma} y^4}{\left(1 - \frac{8}{7!}\frac{\gamma-1}{\gamma} y^4\right)^2 + Z^2} . \tag{25}$$

Next, at $\frac{16}{7!}\frac{\gamma-1}{\gamma} y^4 < 1$ the following inequality is valid:

$$\frac{1 - \frac{8}{7!}\frac{\gamma-1}{\gamma} y^4}{\left(1 - \frac{8}{7!}\frac{\gamma-1}{\gamma} y^4\right)^2 + Z^2} < \frac{1}{1 - \frac{8}{7!}\frac{\gamma-1}{\gamma} y^4} < 1 + \frac{16}{7!}\frac{\gamma-1}{\gamma} y^4 . \tag{26}$$

At $\gamma = 1.4$ the condition $\frac{16}{7!}\frac{\gamma-1}{\gamma} y^4 < 1$ is fulfilled at $y < 5.5$. In this region the condition $X + Z < 1$ is also valid. Therefore at $y < 5.5$ we may write

$$1 - \left(\frac{\gamma-1}{30\gamma}\right)^2 y^4 < \frac{1 - X}{(1 - X)^2 + Z^2} < 1 + \frac{16}{7!}\frac{\gamma-1}{\gamma} y^4 . \tag{27}$$

In the region, where

$$\frac{Ay^2}{1 + B^2 y^2} - 3\left(1 + \frac{c}{y}\right)\left[1 - \left(\frac{\gamma-1}{30\gamma}\right)^2 y^4\right] + \frac{c}{y} < 0 , \tag{28}$$

we have an estimate for β_3:

$$\left|\frac{Ay^2}{1 + B^2 y^2} - 3\left(1 + \frac{c}{y}\right)\left(1 + \frac{16}{7!}\frac{\gamma-1}{\gamma} y^4\right) + \frac{c}{y}\right| > |\beta_3| > \left|\frac{Ay^2}{1 + B^2 y^2} - 3\left(1 + \frac{c}{y}\right)\left[1 - \left(\frac{\gamma-1}{30\gamma}\right)^2 y^4\right] + \frac{c}{y}\right| . \tag{29}$$

At $y < 5.5$ condition (28) is satisfied.

We will write the obvious inequality

$$|\beta_1| = \sqrt{\beta_2^2 + \beta_3^2} < |\beta_3| + \frac{\beta_2^2}{2|\beta_3|}. \tag{30}$$

From the estimates of (14) it can be easily seen that at $y < 5.5$ the following inequality is satisfied a fortiori:

$$|\beta_2| < |\beta_3|, \tag{31}$$

therefore

$$\frac{\partial}{\partial|\beta_3|}\left(|\beta_3| + \frac{\beta_2^2}{2|\beta_3|}\right) = 1 - \frac{\beta_2^2}{2\beta_3^2} > 0. \tag{32}$$

Having omitted cumbersome calculations, when the following inequality is satisfied:

$$6\left(1+\frac{c}{y}\right)\left(\frac{\gamma-1}{30\gamma}\right)^2 y^4 < 3 + \frac{2c}{y} - Ay^2 \tag{33}$$

we have the estimate

$$-\frac{\left(1+\frac{c}{y}\right)\left(\frac{\gamma-1}{\gamma}\right)^2 \frac{y^4}{15} + 8\cdot 10^{-6} + \left(1+\frac{c}{y}\right)\frac{\gamma-1}{\gamma}10^{-3}y^2}{\left(3+2\frac{c}{y}-Ay^2\right)^2}$$

$$< \frac{1}{|\beta_1|} - \frac{1}{3+\frac{2c}{y}-Ay^2} < \frac{6\left(1+\frac{c}{y}\right)\left(\frac{\gamma-1}{30\gamma}\right)^2 y^4}{\left(3+\frac{2c}{y}-Ay^2\right)^2}. \tag{34}$$

In the region with $y < 5.5$, inequality (33) is satisfied. Equation (34) yields

$$\left|\frac{1}{\beta_1}\right| = \frac{1}{3+\frac{2c}{y}-Ay^2}\left\{1 + \eta\,\frac{6\cdot 10^{-3}\left[\left(1+\frac{c}{y}\right)^2 y^4 + 1.5\right]}{3+\frac{2c}{y}-Ay^2}\right\}, \quad |\eta| << 1. \tag{35}$$

At $y < 2.2$

$$S = \frac{6\cdot 10^{-3}\left[\left(1+\frac{c}{y}\right)^2 y^4 + 1.5\right]}{3+2\frac{c}{y}-Ay^2} < 0.06,$$

i.e., the function $\varphi(y) = \dfrac{1}{3+2\frac{c}{y}-Ay^2}$ within the interval (0, 2.2) approximates the function $f(y) = |\beta_1^{-1}|$ with accuracy not less than 6%. Figure 2 depicts the qualitative behavior of the corresponding curves; the function $f(y) = |\beta_1^{-1}|$ is

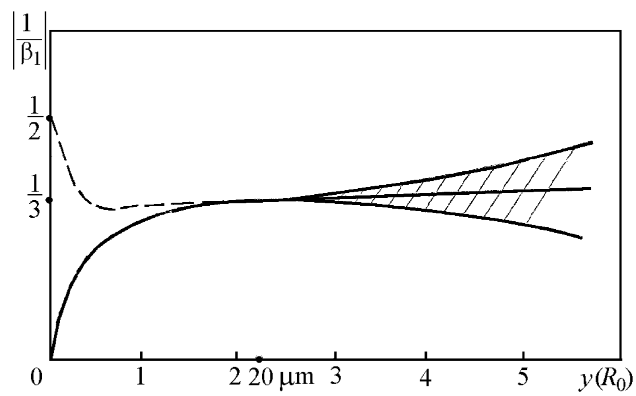

Fig. 2. The qualitative form of the dependence of $|\beta_1^{-1}|$ on R_0.

located in the hatched region. Returning to the variable $R_0 = Gy$ (G = 9.3 μm), we note that in the region with $R_0 \leq 20$ μm the dimensionless amplitude of oscillations $|\alpha| p_\infty/p_a$ is monotonic (resonance is impossible) and limited (in no way able to take the values ~15, as depicted in Fig. 1).

Note that in [1] the dimensionless amplitude $|\alpha| p_\infty/p_a$ is considered as a function of the equilibrium radius R_0, which is incorrect, since the equilibrium pressure in the bubble $p_0 = p_\infty + \frac{2\sigma}{R_0}$ depends on R_0 (at a fixed hydrostatic pressure in the liquid p_∞). It can be easily shown that in this case too in the region with $R_0 \leq 20$ μm resonance is impossible. Moreover, when the initial radius of the bubble tends to zero, $|\beta_1^{-1}| \to 0.5$ rather than to zero, as shown on the curves given in [1]. Indeed, in this case for $R_0 \leq 20$ μm ($y \leq 2.2$)

$$|\beta_1^{-1}| = \frac{1+\frac{c}{y}}{3+2\frac{c}{y}-Ay^2}\left\{1+\eta\frac{6\cdot10^{-3}\left[\left(1+\frac{c}{y}\right)^2 y^4+1.5\right]}{3+2\frac{c}{y}-Ay^2}\right\}. \quad (36)$$

The qualitative behavior of the $|\beta_1^{-1}|$ curve for this case is shown in Fig. 2 by a dashed line.

Conclusions. By means of a detailed analysis the inaccuracy of the principal conclusion drawn in [1] on the existence of two resonant frequencies of a gas bubble of constant mass is shown.

NOTATION

a, velocity of sound, m/sec; b, dimensionless parameter; c_v, c_p, heat capacity at constant volume and pressure, m^2/(sec^2·K); $D_g = \lambda_g/(\rho_0 c_{vg})$, thermal diffusivity of a gas, m^2/sec; $f = \omega/2\pi$, circular frequency, sec^{-1}; F, gas constant, m^2/(sec^2·K); i, imaginary unit; Im, imaginary part; Re, real part; p, pressure, bar; P, dimensionless amplitude of the pressure in a bubble; p_a, amplitude of the acoustic field, bar; r, radial coordinate, m; R, radius of a bubble, m; S, resonance function; t, time, sec; T, temperature, K; V, velocity, m/sec; α, dimensionless amplitude of bubble oscillations; γ, adiabatic exponent; ε_1 and ε_2, small quantities; η, dimensionless variable; θ, dimensionless temperature; λ, thermal conductivity coefficient, kg·m/(sec^3·K); μ, coefficient of viscosity, kg/(m·sec); ρ, density, kg/m^3; σ, surface tension coefficient, kg/sec^2; ω, frequency of oscillations, sec^{-1}. Subscripts: a, acoustic field parameters; ∞, parameters at infinity; 0, parameters in the state of equilibrium; g, parameters of the gas; liq, parameter of the liquid.

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